AN INTRODUCTION TO TRIGONOMETRY

(A NEW OUTLOOK)

by

K. RAMACHANDRA, P.G. VAIDYA and K.G.BHAT

This article ”An Introduction to Trigonometry (A new approach)” is a publication of the Hardy-Ramanujan Society. It was refereed by an excellent mathematician and passed for publication.

To Professors

A.BAKER
R.BALASUBRAMANIAN
K.CHANDRASEKHARAN
K.KASTURIRANGAN
RAGHAVAN NARASIMHAN
RODDAM NARASIMHA
RAJA RAMANNA
A.SANKARANARAYANAN
B.V.SREENKANTAN

for the encouragement.

PREFACE

This is a remarkably readable book authored by Prof. K.Ramachandra, Prof. Prabhakar G. Vaidya and Mr. Kishor Bhat of the National Institute of Advanced Studies, Bangalore. The senior author Prof.K.Ramachandra is one of the most eminent Mathematicians in the World who has the distinction of having Erdös Number 1, which in the mathematical World has very high significance.

This book can be studied at three levels. At one level, it is a book meant for high school students to learn basic ideas in trigonometry. The authors have kept the presentation simple and accessible to high school students. It assumes very little previous knowledge. However, I do think that a certain amount of "mathematical maturity" is needed and therefore it might be useful especially for highly motivated students.

At the second level it deals with the work of two geniuses. One was Euler. In 1748 he discovered a formula which relates the cosine and sine of an angle as the real and imaginary parts of a complex exponential given by

$$e^{i\theta} = \cos\ \theta + i\ \sin\ \theta$$

Richard Feynman, in his famous lectures in physics, called Euler's formula "our jewel" and "the most remarkable formula in mathematics".

Well over a hundred years after Euler, Ramanujan working without an access to most modern mathematics, rediscovered it. This discovery disappointed him when he came to know that Euler had discovered this much earlier. Therefore he refrained from publishing proof of this formula as in the case of most of his work which has intrigued many scholars. We now know from many of his biographies that Ramanujan discovered and rediscovered many amazing results and some of them by his "own method." When questioned by Hardy he could not formally explain his method. This is a very common phenomenon. The working of human brains, (and certainly as pertaining to geniuses), is quite mysterious. However, with a lot of effort and patience many of them explain the intricacies of new methods in simpler terms. Yet, quite often, their students will tell you that this process of understanding is quite intricate. Students need many examples and suddenly they say "oh, now I understand". Unfortunately, this never happened with Ramanujan's method. He stopped working on this method while in Cambridge and no one today knows what his method was. However, we do know from the comments of his widow (Janaki) and his letters that after returning to India he had started working on "his method" again and found many new intriguing results.

So, the mathematical world feels a deep sense of loss about this lost method. Do we have any hope of finding it? This is why I personally found this book quite fascinating when I started reading it. Ever since his childhood Professor Ramachandra has been a devotee of Ramanujan. In fact, his whole academic life is a monument to this devotion. Ramachandra got interested in Ramanujan's work ever since he read the book "Ramanujan - twelve lectures on subjects suggested by his life and works" by G.H.Hardy.

That brings us to Euler's formula. Professor Ramachandra wrote this book primarily asking a question, how would Ramanujan have found this formula. His answer is this book. If there is anyone today who might understand how Ramanujan's mind worked, it would be Professor Ramachandra or - and this is my challenge - some young student who loves mathematics as much as Ramanujan and studies this book and may be she would say "oh, now I see a whole new way of solving problems!"

At the third level, this book should be taken seriously by those who are responsible for setting curricula for high school studies. I believe that the introduction of complex numbers and their applications is delayed more than necessary. My other colleague and one of the authors of this book Professor Vaidya often says that things become very simple once you recognize that they are "complex". The complex numbers are introduced much too late in our studies. A lot of results of plain geometry, for example, become much simpler in the context of the complex plane.

In this book the authors present what could possibly have been Ramanujan's method of deriving Euler's formula on sines and cosines. Everything starts from the relation $(a+b)^2 = a^2 + b^2 + 2ab$ which leads to pythagoras theorem and so on, as demonstrated in this book. It should be mentioned that the method is very different from that has existed in text books, so far.

CHAPTER - I

PYTHAGORAS' THEOREM

We give two proofs of this Theorem one based on $(a+b)^2 = a^2 + 2ab + b^2$ and another on $(a-b)^2 = a^2 - 2ab + b^2$. (Both these identities are valid for any two complex numbers a, b).

<u>FIRST PROOF</u> (using $(a+b)^2 = a^2 + 2ab + b^2$). Consider the triangle Δ SRC, where $\hat{C}$ is a right angle. Draw the square PQRS. Draw Δ QBR in such a way that $CR = QB$ and $SC = RB$. Similarly draw the triangles Δ PAQ and Δ PDS. All the four triangles are congruent (since three sides of one equal three sides of the other). Each of these triangles is equal in area to that of Δ SCR, which is plainly equal to $\frac{1}{2}\ CR \times SC$. The four triangles together make up $4(\frac{1}{2} CR \times SC) = 2CR \times SC$. Also they are equal to $(CD)^2 - (SR)^2$. Thus (since $CD = DS + SC = (CR + SC)$),

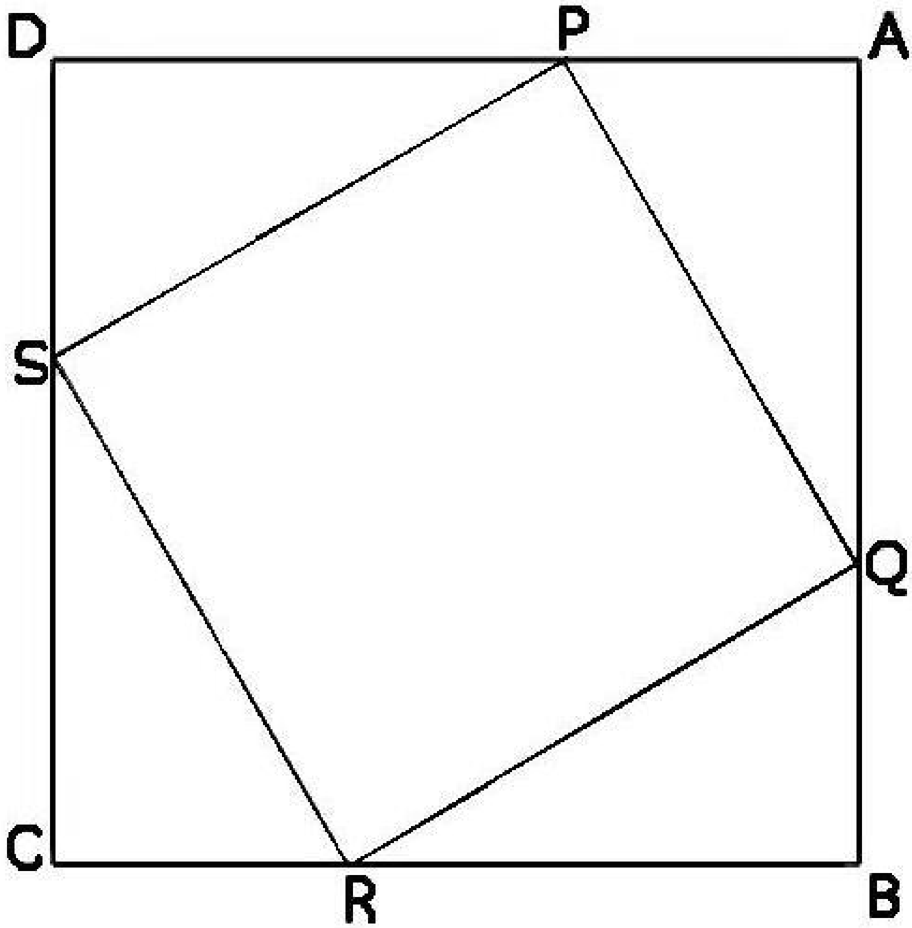

Figure 1

$$2CR \times SC = (CR + SC)^2 - (SR)^2.$$

Hence

$$\begin{aligned} (SR)^2 &= (CR)^2 + 2CR \times SC + (SC)^2 - 2CR \times SC \\ &= (CR)^2 + (SC)^2. \end{aligned}$$

SECOND PROOF(using $(a-b)^2 = a^2 - 2ab + b^2$). Consider the triangle ΔABP right angled at $\hat{P}$. Construct a triangle ΔQBC ($\hat{Q}$ = a right angle) such that $BQ = AP$ and $CQ = BP$. Plainly $PQ = BP - AP$ (since ΔABP is congruent to ΔQBC). Certainly $AB = BC$. Do the same thing for triangle ΔRDC and ΔADS. The four small triangles make up an area of $4(\frac{1}{2}AP \times BP) = 2AP \times BP$. Hence

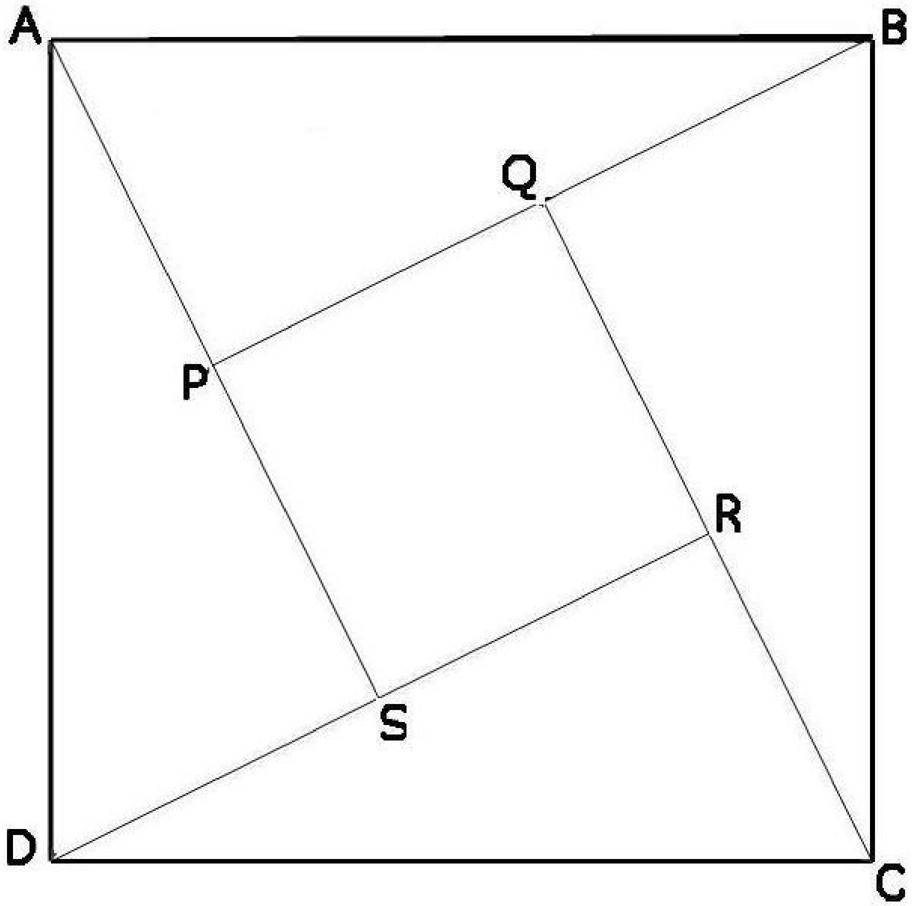

Figure 2

$$(AB)^2 = (BP - AP)^2 + 2AP \times BP$$

$$= (BP)^2 + (AP)^2,$$

using $(a-b)^2 = a^2 - 2ab + b^2$.

<u>**REMARK**</u> We have used the fact that the area of a right angled triangle is equal to half the product of the sides containing the right angle. We will amplify this remark again in chapter III.

CHAPTER - II

SIMILAR TRIANGLES THEOREM

In this chapter we prove that PYTHAGORAS' theorem implies that if two triangles are similar (in other words if three angles of one are equal to three angles of the other) then their corresponding sides are proportional. In order to prove this we can not do any thing better than reproducing from the reference [3]. We deduce the general case from the following theorem as a corollary.

THEOREM 1 *If two right angled triangles are similar then their corresponding sides are proportional.*

To prove the corollary, divide each of the similar triangles PQR and XYZ into right angled triangles as shown in figure 1 and apply theorem 1 twice

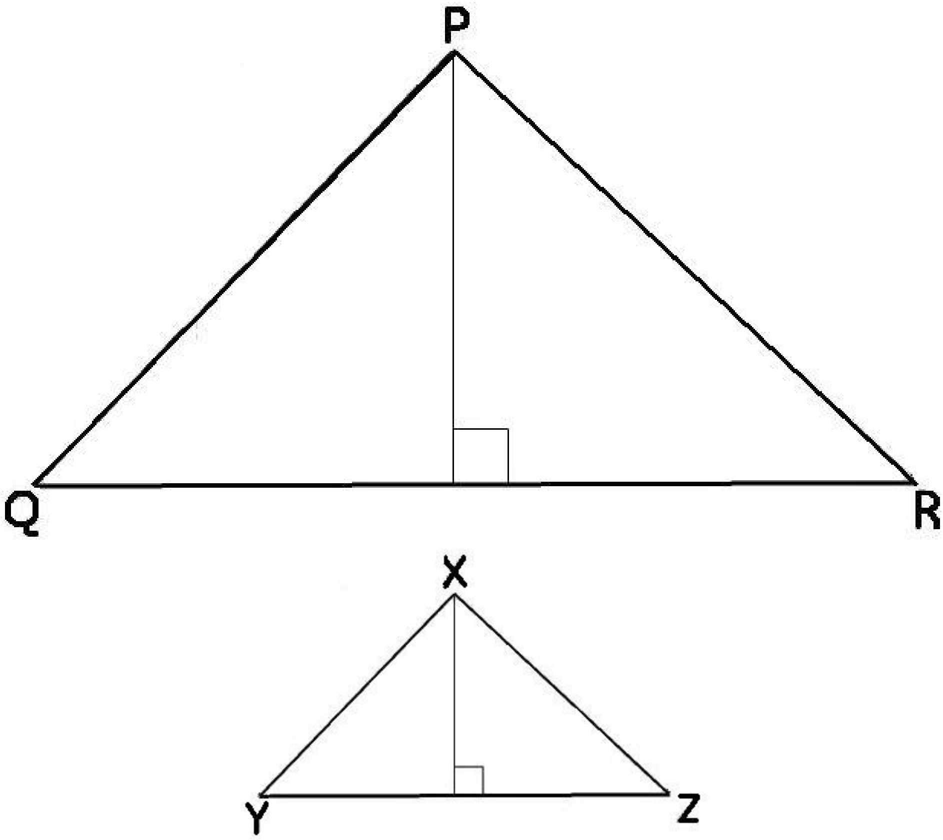

Figure 1

There is a well-known method of proving PYTHAGORAS' Theorem using theorem 1, but it is less well known that deduction can be made in the opposite direction namely: we can prove theorem 1 using PYTHAGORAS' theorem. The following proof may be new.

Let ABC and DEF be similar right angled triangles. If they are congruent then there is nothing to prove, so assume without loss of generality that DE is less than AB. Superimpose ΔDEF on ΔABC in such a way that D coincides with A and DE, DF fall respectively on AB, AC as in figure 2. Draw EG perpendicular to BC.

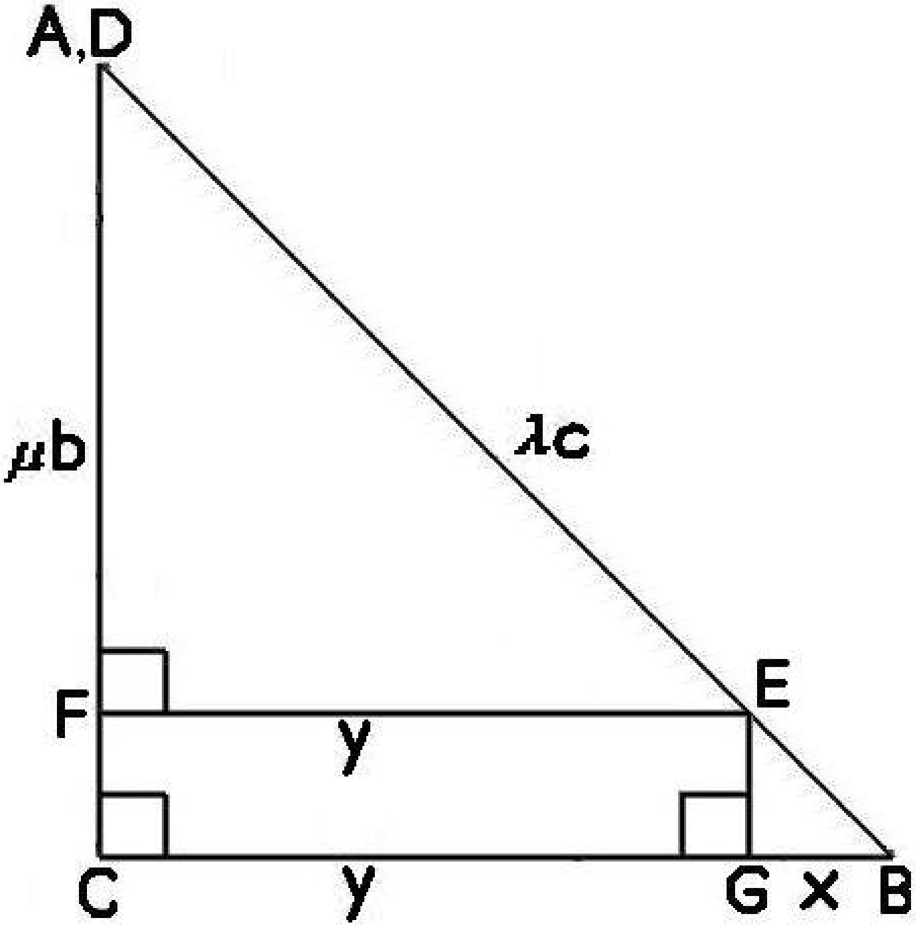

Figure 2

Write $AB = c, BC = a, CA = b, AE = \lambda c, AF = \mu b$, where $0 < \lambda < 1$ and μ is positive. It suffices to show that $\lambda = \mu$. Since E lies between A and B, F cannot lie outside AC, for otherwise the parallel lines EF and BC would intersect. Hence $0 < \mu < 1$.

Also write $BG = x, GC = EF = y$. Applying Pythagoras' theorem we have

$$x^2 = (c - \lambda c)^2 - (b - \mu b)^2, \; y^2 = \lambda^2 c^2 - \mu^2 b^2 \text{ and } a^2 = c^2 - b^2. \tag{1}$$

Now $x = a - y$, so $x^2 = a^2 + y^2 - 2ay$. Hence $2ay = a^2 + y^2 - x^2$. Substituting from (1) we obtain

$$2ay = 2\lambda c^2 - 2\mu b^2$$

Hence

$$a^2 y^2 = (\lambda c^2 - \mu b^2)^2.$$

Substituting again for a^2 and y^2 from (1) we obtain after simplification $(\lambda - \mu)^2 b^2 c^2 = 0$ and hence $\lambda = \mu$ as required.

REMARK The proof of the theorem on similar triangles reproduced here is a simplified version (due to the referee) of the paper [3] by K.Ramachandra.

CHAPTER - III

ADDITION THEOREM FOR SINES and COSINES

From the results of chapter II it is clear that the ratios

$$sin\ \theta = \frac{opposite\ side}{hypotenuse}\ and\ cos\ \theta = \frac{adjacent\ side}{hypotenuse}$$

are quantities which do not depend on the right angled triangle but only on the angle θ. The same is true of

$$\tan\theta = \frac{opposite\ side}{adjacent\ side},\quad cot\ \theta = (\tan\ \theta)^{-1}$$

and other ratios cos sec $\theta = (sin\theta)^{-1}, sec\ \theta = (cos\ \theta)^{-1}$.

It is very surprising that these ratios have an addition theorem.

ADDITION THEOREM *We have*

$$sin(\alpha+\beta) = sin\ \alpha\ cos\ \beta + cos\ \alpha\ sin\ \beta$$

and

$$cos(\alpha+\beta) = cos\ \alpha\ cos\ \beta - sin\ \alpha\ sin\ \beta$$

where α, β *and* $\alpha+\beta$ *are positive angles (all less than a right angle).*

COROLLARY 1. *Let* $i = \sqrt{-1}$. *Then*

$$\begin{aligned} & cos(\alpha+\beta) + i\ sin(\alpha+\beta) \\ = \ & cos\ \beta(cos\ \alpha + i\ sin\ \alpha) + sin\ \beta(i\ cos\ \alpha - sin\ \alpha) \\ = \ & cos\ \beta(cos\ \alpha + i\ sin\ \alpha) + i\ sin\ \beta(cos\ \alpha + i\ sin\ \alpha) \\ = \ & (cos\ \alpha + i\ sin\ \alpha)\ (cos\ \beta + i\ sin\ \beta) \end{aligned}$$

COROLLARY 2. *Let* θ *be any angle. Then for all positive integers* n

$$cos\ \theta + isin\ \theta = \left(cos\ \frac{\theta}{n} + i\ sin\frac{\theta}{n}\right)^n$$

(Note: RH makes sense whatever θ *(positive angle) provided* n *is large.* Also if we take $cos^2\theta + sin^2\ \theta = 1$ to be valid for negative θ and interpret $cos(-\theta) = cos\ \theta$ and $sin(-\theta) =$

$-sin\ \theta$ we will have $\cos^2\theta + sin^2\ \theta = 1 = (cos\ \theta + i\ sin\ \theta)(cos\ (-\theta) + i\ sin\ (-\theta))$. Hence we can uphold corollary 2 to negative integers n also.

PROOF OF THE MAIN THEOREM (Proof of addition theorem for $sin\ \theta$, due to I.M.Gelfand)

We begin with

LEMMA 1. *Area of any triangle is equal to half the product of any two sides multiplied by the sine of the included angle.*

PROOF Suppose the triangle is ΔABC. Draw AD perpendicular to BC. Area of ΔABC is equal to the sum of the areas of ΔABD and ΔADC

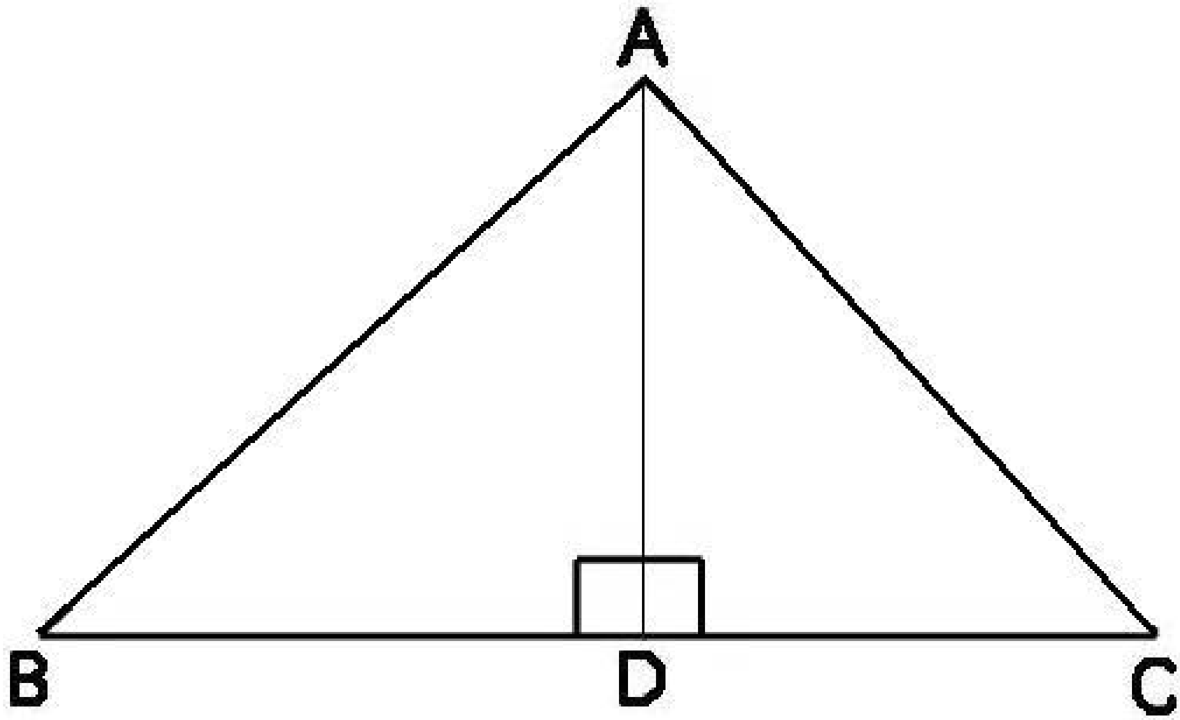

Figure 1

$$= \frac{1}{2}BD \times AD + \frac{1}{2}DC \times AD$$
$$= \frac{1}{2}AD(BD + DC) = \frac{1}{2}AD \times BC.$$

But $AD = AB\times$ sine of the angle $A\hat{B}D$. Hence area of $\Delta ABC = \frac{1}{2}AB \times BC\times$ sine of the angle $A\hat{B}D$ and this proves the lemma.

REMARK. We have used the fact that the area of a right-angled triangle (say ΔACD) is $\frac{1}{2}DC \times AD$ which is half of the area of the rectangle APCD (see the figure 2) since ΔACD is congruent to ΔAPC.

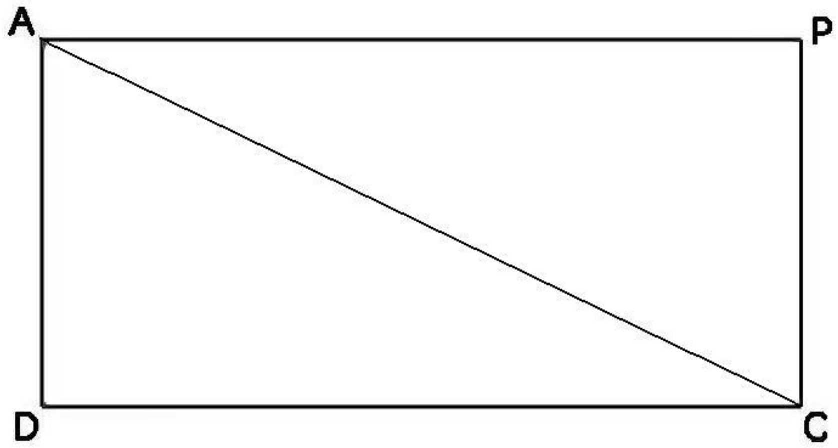

Figure 2

Consider the figure 3, where angle $B\hat{A}D = \alpha$ and $D\hat{A}C = \beta$, and AD is perpendicular to BC. Select B such that angle $B\hat{A}D = \alpha$ and C such that $D\hat{A}C = \beta$. Plainly by $B\hat{A}D = \alpha + \beta$. Area of $\Delta BAC = \frac{1}{2}AB \times AC\ sin(\alpha + \beta)$. Also area of $\Delta BAC =$ the sum of the areas ΔBAD and ΔDAC.

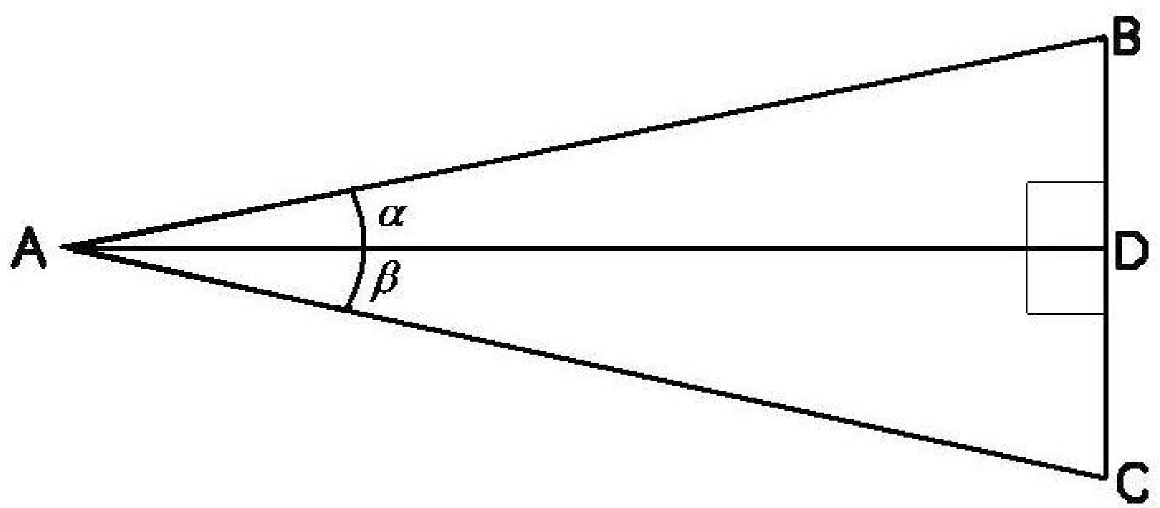

Figure 3

$$= \frac{1}{2}AD \times BD + \frac{1}{2}AD \times DC.$$

Hence $AB \times AC \ sin(\alpha+\beta) = AD \times BD + AD \times DC$. But $AD = AC \ cos \ \beta = AB \ cos \ \alpha$ and $BD = AB \ sin \ \alpha$ and also $DC = AC \ sin\beta$.

There follows

$$AB \times AC \ sin(\alpha+\beta) = AC \ cos \ \beta \times AB \ sin \ \alpha + AB \ cos \ \alpha \times AC \ sin \ \beta.$$

Cancelling $AB \times AC$ throughout we obtain

$$sin(\alpha+\beta) = sin \ \alpha \ cos \ \beta + cos \ \alpha \ sin \ \beta.$$

Next (using $cos \ \frac{\pi}{4} = sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ which follows from a consideration of right angled isosceles triangle)

$$\begin{aligned}
cos \ \theta &= sin\left(\frac{\pi}{4}+\frac{\pi}{4}-\theta\right) \\
&= sin\frac{\pi}{4}cos \ \left(\frac{\pi}{4}-\theta\right) + cos \ \frac{\pi}{4}sin\left(\frac{\pi}{4}-\theta\right) \\
&= sin\frac{\pi}{4}sin\left(\frac{\pi}{4}+\theta\right) + cos \ \frac{\pi}{4}sin\left(\frac{\pi}{4}-\theta\right) \\
&= sin\frac{\pi}{4}\left(sin\frac{\pi}{4}cos \ \theta + cos \ \frac{\pi}{4}sin \ \theta\right) + cos \ \frac{\pi}{4}sin\left(\frac{\pi}{4}-\theta\right) \\
&= \frac{1}{2}cos \ \theta + \frac{1}{2}sin \ \theta + \frac{1}{\sqrt{2}}sin\left(\frac{\pi}{4}-\theta\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{\sqrt{2}}sin\left(\frac{\pi}{4}-\theta\right) &= \frac{1}{2}cos \ \theta - \frac{1}{2}sin \ \theta \\
i.e, \ sin\left(\frac{\pi}{4}-\theta\right) &= \frac{1}{\sqrt{2}}(cos \ \theta - sin \ \theta).
\end{aligned}$$

From this we obtain (by using $cos \ \theta = sin(\frac{\pi}{2}-\theta)$ and $\sin \theta = \cos \ (\frac{\pi}{2}-\theta)$)

$$\begin{aligned}
cos \ (\theta+\phi) &= sin\left(\frac{\pi}{4}-\theta+\frac{\pi}{4}-\phi\right) \\
&= sin(\frac{\pi}{4}-\theta)cos \ (\frac{\pi}{4}-\phi) + cos \ (\frac{\pi}{4}+\theta)sin(\frac{\pi}{4}-\phi) \\
&= sin(\frac{\pi}{4}-\theta)sin(\frac{\pi}{4}+\phi) + sin(\frac{\pi}{4}+\theta)sin(\frac{\pi}{4}-\phi)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}(cos\ \theta - sin\ \theta)\frac{1}{\sqrt{2}}(cos\ \phi + sin\ \phi) + \frac{1}{\sqrt{2}}(sin\ \theta + cos\ \theta)\frac{1}{\sqrt{2}}(cos\ \phi - sin\ \phi) \\
&= \frac{1}{2}(cos\ \theta\ cos\ \phi + cos\ \theta\ sin\ \phi - sin\ \theta\ cos\ \phi - sin\ \theta\ sin\ \phi) \\
&+ \frac{1}{2}(sin\ \theta\ cos\ \phi - sin\ \theta\ sin\ \phi + cos\ \theta\ cos\ \phi - cos\ \theta\ sin\ \phi) \\
&= cos\ \theta\ cos\ \phi - sin\ \theta\ sin\ \phi
\end{aligned}
$$

which is the addition theorem for $cos\ \theta$ namely $cos\ (\theta + \phi) = cos\ \theta\ cos\ \phi - sin\ \phi\ sin\ \phi$. i.e. $cos(\alpha + \beta) = cos\ \alpha\ cos\ \beta - sin\ \alpha\ sin\ \beta$ with a change of notation. This proves our addition theorem completely. It must be remembered that $0 \leq \alpha \leq \frac{\pi}{2}$ and $0 \leq \beta \leq \frac{\pi}{2}$ $0 \leq \alpha + \beta \leq \frac{\pi}{2}$. But we extend it to other (real) values of α and β by using

$$(cos\ \theta + i\ sin\ \theta) = \left(cos\ \frac{\theta}{n} + i\ sin\ \frac{\theta}{n}\right)^n$$

valid for all integers $n \neq 0$.

CHAPTER - IV

RADIAN MEASURE AND CALCULATION OF TRIGONOMETRIC RATIOS

Although we start with sexagesimal measure such as $30^o, 45^o, 180^o$ and so on, we find it convenient to designate by 2π the circumference of a circle of unit radius. We call the angle around a point 2π radians. We divide the circumference into very small equal parts say k parts and each part is of length $\frac{2\pi}{k}$. The angle subtended at the center by each part is $\frac{2\pi}{k}$ radians. By choosing k large, we can define (by rule of three) the angle $\theta (0 < \theta \leq 2\pi)$ subtended at the center by arc of length θ will be θ radians. Of course we can calculate π by using

$$tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

and putting $x = 1$ and $\tan^{-1} 1 = \frac{\pi}{4}$ (radian measure). These things will be worked out in Chapter VI. For the calculation of trigonometric ratios we start with

LEMMA 1. *For* $0 < \theta \leq \frac{\pi}{2}$, *we have*

$$cos\ \theta = 1 + O(\theta^2)$$

NOTE From now on we employ the notation $O(\ldots)$ to mean "less than a constant times ...". Thus stated in other words the lemma reads

$$|cos\ \theta - 1| \leq C\theta^2$$

where $C > 0$ is an absolute constant. From now on, all angles will be in radian measure.

PROOF. $|cos\ \theta - 1| = 1 - cos\ \theta = \frac{1 - cos^2\ \theta}{1 + cos\ \theta} \leq (sin\ \theta)^2 \leq \theta^2$ and we prove a more precise result regarding $(sin\ \theta)$ in the next lemma.

LEMMA 2. $|sin\ \theta - \theta| = O(\theta^3)$,

PROOF. In the figure 1, $AB = AC = AF = 1, B\hat{A}C = \theta =$ arc BFC, and AGF is perpendicular to BC. DFE is tangent to arc BFC touching it at F. Also F is the middle point of arc BFC. We recall that the area of any triangle is half the product of any two sides multiplied by the sine of the included angle. Hence by dividing the arc θ into small bits area of sector ABFC=θ. Area of ΔABC is $\frac{1}{2}\ sin\ \theta$.

Again $BD = GF = AF - AG = 1 - cos\ \frac{\theta}{2}, BC = 2sin\frac{\theta}{2}$. Hence

$$0 < \frac{1}{2}\theta - \frac{1}{2}sin\ \theta \leq \left(2\ sin\frac{\theta}{2}\right)\left(1 - cos\ \frac{\theta}{2}\right) \leq \left(2\ sin\frac{\theta}{2}\right)\left(1 - cos^2\ \frac{\theta}{2}\right)\left(1 + cos\ \frac{\theta}{2}\right)^{-1}$$

$$\leq \theta \left(sin\frac{\theta}{2} \right)^2 \leq \frac{1}{4}\theta^3.$$

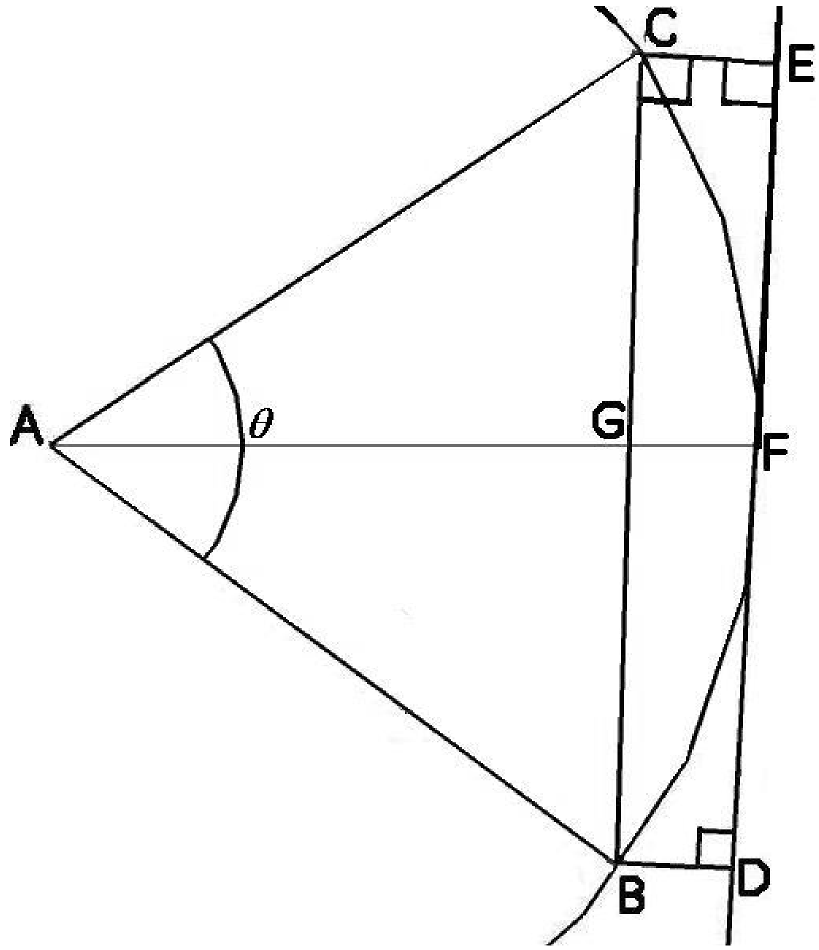

Figure 1

Thus $0 < \theta - sin\ \theta \leq \frac{1}{2}\theta^3$. This proves the lemma. Our next step is

THEOREM *If* θ *is in radian measure then*

$$(cos\ \theta + i\ sin\ \theta) = \left(1 + \frac{i\theta}{n}\right)^n + O\left(\frac{1}{n^2}\right).$$

PROOF. We begin with the identity

$$\begin{aligned} & A^n - B^n \\ = \ & (A-B)(A^{n-1} + A^{n-2}B + A^{n-3}B^2 + \ldots + B^{n-1}) \\ \leq \ & |A-B|(nJ^{n-1}) \end{aligned}$$

where $J = \max(|A|, |B|)$. We observe

$$\begin{aligned}
cos\ \theta + i\ sin\ \theta &= \left(cos\ \frac{\theta}{n} + i\ sin\frac{\theta}{n}\right)^n \text{ and also} \\
\left(cos\ \frac{\theta}{n} + i\ sin\frac{\theta}{n}\right)^n - \left(1 + \frac{i\theta}{n}\right)^n &= A^n - B^n, \left(\text{where A} = \cos\frac{\theta}{\text{n}} + \text{i} \sin\frac{\theta}{\text{n}} \text{ and B} = 1 + \frac{\text{i}\theta}{\text{n}}\right), \\
|A^n - B^n| \leq |A - B| n J^n \text{ where J} &= \max\left(\left|cos\frac{\theta}{n} + i\ sin\frac{\theta}{n}\right|, \left|1 + \frac{i\theta}{n}\right|\right) \leq 1 + \frac{C_o}{n}
\end{aligned}$$

(where C_o is a positive and constant, and so $nJ^n \leq n(1 + \frac{C_o}{n})^n \leq Dn$, where D is a positive constant. The inequality $(1 + \frac{C_o}{n})^n \leq D$ for all large n will be proved in the next chapter).

$$\begin{aligned}
|A - B| &= \left|cos\frac{\theta}{n} - 1 + i\left(sin\frac{\theta}{n} - \frac{\theta}{n}\right)\right| \\
&\leq \left|cos\frac{\theta}{n} - 1\right| + \left|sin\frac{\theta}{n} - \frac{\theta}{n}\right| \\
&= O\left(\frac{1}{n^2} + \frac{1}{n^3}\right) \\
&= O\left(\frac{1}{n^2}\right), \text{by using lemmas 1 and 2.}
\end{aligned}$$

Thus

$$\begin{aligned}
& cos\ \theta + i\ sin\theta - \left(1 + \frac{i\theta}{n}\right)^n \\
= & \left(cos\frac{\theta}{n} + i\ sin\frac{\theta}{n}\right)^n - \left(1 + \frac{i\theta}{n}\right)^n \\
= & O\left(D\ .n\ .\frac{1}{n^2}\right) = O\left(\frac{1}{n}\right)
\end{aligned}$$

and this proves the theorem.

LEMMA *We have the inequality*

$$\sum_{n=0}^{\infty} \frac{C^n}{n!} \leq \sum_{0 \leq n \leq 100C^3} \frac{C^n}{n!} + 4$$

where C is any positive constant.

PROOF Put $K = 100C^3$ (without loss of generality we can assume that C is a positive integer). We can ignore $\sum_{0 \leq n \leq |K} \frac{C^n}{n!}$. Now

$$\sum_{n \geq K+1} \frac{C^n}{n!} \leq \sum_{n \geq K+1} \frac{C.C \ldots, C \textit{ to } n \text{ terms}}{100C^3.100C^3 \ldots \textit{ to } n \text{ terms}}$$

$$\leq \sum_{n\geq K+1} \frac{C^2.C^2.\ldots.C^2\ to[\frac{n}{2}]+1\text{ terms}}{100C^3.100C^3\ldots to[\frac{n}{2}]+1\text{ terms}}$$

$$\leq \sum_{n\geq K+1} \frac{1}{100C.100C\ldots[\frac{n}{2}]+1\text{ terms}}$$

$$\leq \sum_{n=0}^{\infty} 2.2^{-n} = 4. \text{ since } 100\mathrm{C} \geq 4 \text{ and}$$

$$4.4.\ldots\text{to } [\frac{\mathrm{n}}{2}]+1 \text{ terms } \geq 2.2^{-\mathrm{n}}.$$

Letting $n\to\ \infty$ we have

COROLLARY. $\lim_{n\to\infty}\ \left(1+\frac{i\theta}{n}\right)^n$ *exists and is equal to* $cos\ \theta + i\ sin\ \theta, \theta$ *being in radian measure.*

REMARK. The notion of a limit (and consequently the concept of convergence of a sequence (such as infinite series, infinite products and infinite continued fractions) depends on the notion of a distance of a real or a complex number from the origin). This necessitates the notion of the real line or the complex plane as the case may be. The distance of real number a is $|a|$ and that of a complex number $x+iy$ is the positive square root $\sqrt{x^2+y^2}$.

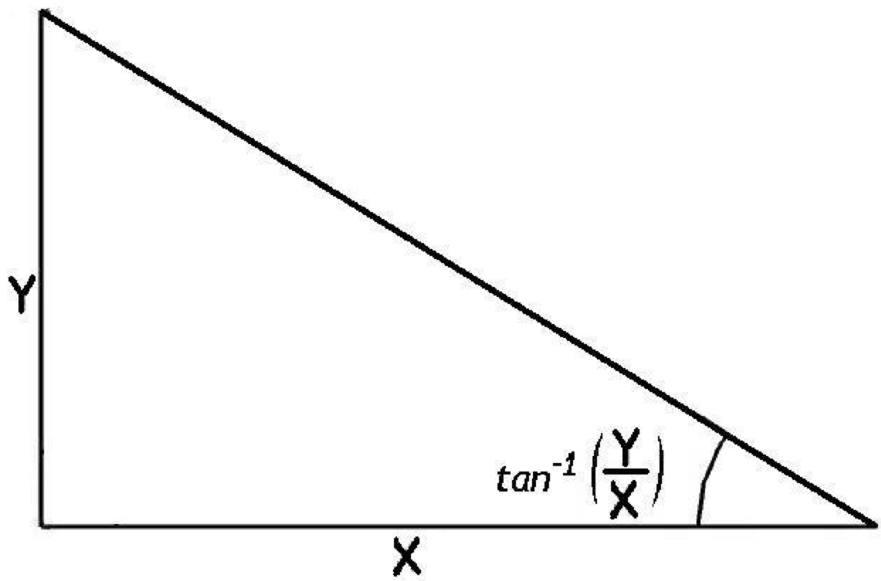

Figure 2

Also

$$x+iy \quad = \quad \sqrt{x^2+y^2}\ e^{i\ tan^{-1}\frac{y}{x}+2\pi ik}$$

$$= \sqrt{x^2+y^2}\ \left\{\frac{x}{\sqrt{x^2+y^2}} + \frac{iy}{\sqrt{x^2+y^2}}\right\}.$$

Hence

$$\log(x+iy) = \log\sqrt{x^2+y^2} + \mathrm{i}\tan^{-1}\frac{y}{x} + 2\pi ik$$
$$k = 0, \pm 1, \pm 2, \ldots$$

.

These are in some ways a natural way of introducing the distance function (also called Archemedian valuation). There are other ways (called non-Archemedian valuations). For example

$$|2^n|_2 = 2^{-n}\ and\ |6^n|_2 = 2^{-n}.$$

You will surely raise your eyebrows if I say that the distance of 2^n from the origin is 2^{-n} and so 2^n tends to zero as $n \to \infty$. But you need not. There is a rich subject called "p-adic analysis". Every rational number has a p-adic distance called p-adic valuation" associated with every prime p. For example if $(p,6) = 1$ then $|6|_p = 1$. Also we can do differentiation, integration theory of analytic functions and so on. In this theory, the radius of convergence of $\sum_{n=1}^{\infty} \frac{z^n}{n!}$ is not infinity but $p^{-\delta}, \delta = \frac{1}{p-1}$, !!!.

This is certainly not perverted intelligence. The extra-ordinary results of A.Baker have been extended to p-adic valuations. These give very rich dividends to ordinary Diophantine equations and more general Diophantine problems. The best Indian expert on p-adic analysis is Professor T.N. Shorey. He has worked on p-adic transcendence and application of Baker's work to Diophantine questions. From now on we use only the ordinary distances of real and complex numbers from the origin.

In the next chapter we will express $\lim_{n\to\infty}(1+\frac{i\theta}{n})^n$ as an infinite power series in powers of $i\theta$. Separating real and imaginary parts we obtain series for the $sin\theta$ and $cos\ \theta$ in powers of θ.

CHAPTER - V

EXPONENTIAL SERIES

The object of this chapter is to prove the following theorem.

THEOREM 1 *For any fixed complex number $z = x + iy$ we have*

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots.$$

PROOF. For proving this theorem we need the expansion

$$\left(1 + \frac{z}{n}\right)^n = 1 + \frac{n}{1!}\left(\frac{z}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{z}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{z}{n}\right)^3 + \dots$$

to $n+1$ terms, where n is a positive integer (this could have been proved by Ramanujan).

MOTIVATION We can start with

$$(1-x)^{-1} = 1 + x + x^2 + \dots.$$

differentiate both sides with respect to x n times. We get successively (for LHS)

$$1!(1-x)^2,\ 2!(1-x)^{-3},\ 3!(1-x)^{-4}, \dots \textit{ to } n \textit{ terms}$$

RHS will be

$$1.x^0 + 2.x^1 + 3x^2 + 4x^3 + \dots$$
$$2.1.x^0 + 2.3x^1 + 3.4x^2 + \dots$$

We can guess what happens at the r^{th} stage and we get an expansion for

$$(1-x)^{-r}.$$

Here we can replace r by $-r$ and x by $-x$ and thus we can get a formula for

$$(1+x)^n.$$

From this we can get the expansion for $\left(1 + \frac{z}{n}\right)^n$, (which is the well known Binomial theorem for a positive integral index) stated earlier.

We need a lemma

LEMMA

$$\left|\frac{n(n-1)}{n^2} - 1\right| \le \frac{2^2}{n},\ \left|\frac{n(n-1)(n-2)}{n^3} - 1\right| \le \frac{2^3}{n},$$

$$\left|\frac{n(n-1)(n-2)(n-3)}{n^4} - 1\right| \le \frac{2^4}{n}, \dots$$

<u>PROOF</u> We have for $0 < r < n$

$$\left|\frac{n(n-1)(n-2)\dots(n-r)}{n^{r+1}} - 1\right|$$
$$\le \left|\frac{(n-1)(n-2)\dots(n-r)}{n^r} - 1\right|$$

(Note that the first term is positive and less than 1 and so the quantity in question is)

$$\le \left|\left(\frac{n-r}{n}\right)^r - 1\right| = n^{-r}\left(n^r - (n-r)^r\right)$$
$$= n^{-r}\left(n-(n-r)\right)\left(\sum_{j=0}^{r-1} n^j\,(n-r)^{r-1-j}\right)$$
$$\le n^{-r}.r.r\ n^{r-1} = \frac{r^2}{n} = \frac{r^2}{2}.2.\frac{1}{n} \le \frac{2^{r+1}}{n}$$
$$\text{since } 2^{\mathrm{r}} \ge 1 + \mathrm{r} + \frac{\mathrm{r}(\mathrm{r}-1)}{2} = 1 + \frac{\mathrm{r}^2+\mathrm{r}}{2} \ge \frac{\mathrm{r}^2}{2}.$$

This proves the lemma.

Now

$$\left|\left(1+\frac{z}{n}\right)^n - \left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots\right)\right|$$
$$\le \left|\left(1+\frac{|z|}{n}\right)^n - \sum_{m=0}^{n}\frac{|z|^m}{m!}\right| + \sum_{m>n}\frac{|z|^m}{m!}$$
$$\le \left(\frac{|z|^2}{2!}\ \frac{2^2}{n} + \frac{|z|^3}{3!}\ \frac{2^3}{n} + \frac{|z|^4}{4!}\ \frac{2^4}{n} + \dots + \frac{|z|^n}{n!}\ \frac{2^n}{n}\right)$$
$$+ \sum_{m>n}\frac{|z|^m}{m!} = P + Q \textit{ say}.$$

It is clear that

$$Q \le \frac{|z|^n}{n!}\sum_{m=n+1}^{\infty}\frac{|z|^{m-n}n!}{m!}$$
$$= \frac{|z|^n}{n!}\sum_{r=1}^{\infty}\frac{|z|^r n!}{(n+r)!} = \frac{|z|^n}{n!}\sum_{r=1}^{\infty}\frac{|z|^r}{r!}$$

since for $r \ge 1$ we have $\frac{n!}{(n+r)!} \le \frac{1}{r!}(\frac{(n+r)r!}{n!}$ being a binomial coefficient in $(1+x)^{n+r}$ and hence an integer). Hence if $|z| \le C$(we have for some $C > 2$)

$$Q \ \le\ \frac{C^n}{n!}\sum_{r=1}^{\infty}\frac{C^r}{r!} \ \le \frac{C^n}{n!} \textit{ times a constant.}$$

Hence as $n \to \infty,\ Q \to 0$ as is evident from

$$\frac{C^n}{n!} \ \leq \ \frac{C^n}{[\frac{n}{2}] \dots [n]} \leq \left(\frac{C^2}{\frac{n}{2}-1}\right)^{\frac{n}{2}}$$

valid for $n \geq 100C^2$.

Again

$$\begin{aligned} P \ &\leq \ \frac{1}{n}\left\{\frac{|2z|^2}{2!}+\frac{|2z|^3}{3!}+\dots\right\} \\ &\leq \ \frac{1}{n}\sum_{m=1}^{\infty}\frac{|2z|^m}{m!} \\ &\leq \ \frac{1}{n}\sum_{m=1}^{\infty}\frac{|2C|^m}{m!} \text{ for } |\text{z}| \leq \text{C} \end{aligned}$$

It is not hard to prove that the last infinite sum is bounded by a constant depending on C (see the remark before chapter V).

Hence $P \to 0, Q \to 0 \ as \ n \to \infty$, and this proves the theorem.

<u>**REMARK**</u>. We do not go in detail to the theory of infinite series. For our purposes an infinite series of complex terms $\sum_{n=1}^{\infty} a_n$ represents a complex number if it is "convergent". For our purposes a series is said to be convergent if the tail portion

$$\sum_{n=m}^{\infty} \ a_n$$

tends to zero as $m \to \infty$. More precisely if $\left|\sum_{n=m}^{m+N}\ a_n\right|$ tends to zero as $m \to \infty$ whatever $N \geq 1$. In our case

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

is convergent. (In fact it can be differentiated term by term any number of times). Call this $e(z)$.

$$e\left(z_1\right) e\left(z_2\right) = \lim_{n\to\infty}\ \left\{\left(1+\frac{z_1}{n}\right)^n\left(1+\frac{z_2}{n}\right)^n\right\}$$

and the limit can be seem to be

$$\lim_{n\to\infty}\ \left\{1+\frac{z_1+z_2}{n}\right\}^n = e\left(z_1+z_2\right)$$

and so

$$\left(e\left(z\right)\right)^n = e\left(nz\right), \left(e\left(1\right)\right)^z = \lim_{n\to\infty}\ \left\{\left(1+\frac{1}{n}\right)^{zn}\right\}$$

$$= \lim_{n\to\infty} \left\{ \left(1 + \frac{z}{n}\right)^n . \right\}. \text{ Hence } e(z) = e(1)^z, \; e(1)$$

is usually denoted by e.

Thus we have the following theorem.

THEOREM 2 *We have* $\cos\theta + i\sin\theta = \lim_{n\to\infty}(1+\frac{i\theta}{n})^n = e^{i\theta} = 1 + i\theta + \frac{(i\theta^2)}{2!} + \frac{(i\theta)^3}{3!} + \ldots$ *whereby*

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} - + \ldots$$

and

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \ldots$$

REMARK. If a and z are complex number we have to interpret a^z as $\exp\{z \log a\}$. Since $\log a_1 = \log a_2$ will happen with $a_1 = a_2 e^{2\pi i k}$ for any integer k, we have to specify the logarithm. In case a is a positive real number $\log a$ is in uniquely defined (in practice) as the unique real solution of $e^b = a$. But even in this case $\log a$ is in general any of the numbers $b + 2ki\pi (k = 0, \pm 1, \pm 2, \ldots,)$. If $a (\neq 0)$ is complex we write $\frac{a}{|a|} = e^{i\theta}$ (note the LHS has absolute value 1.) If $A + iB$ is of absolute value 1 so its square $A^2 + B^2 = 1$, where A and B are real.

CHAPTER - VI

LOGARITHMIC SERIES

We have seen that

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = 1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots(=e^z.$$

Let $1-x(|x|<1)$ be the series on the RHS. It is possible to invert and find an expression for z in terms of x. Put $(1+\frac{z}{n})^n = 1-x$. There follows

$$\lim_{n\to\infty} n\left((1-x)^{\frac{1}{n}}-1\right) = z.$$

If we use binomial theorem for the (non-integral) index $\frac{1}{n}$, we have

$$\begin{aligned} z &= \lim_{n\to\infty} n\left(-x.\frac{1}{n}+\frac{(-x)^2}{2!}.\frac{1}{n}\ (\frac{1}{n}-1)+\frac{-x^3}{3!}.\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)+\ldots\right) \\ &= \lim_{n\to\infty}(-x-\frac{x^2}{2}-\frac{x^3}{3}+\ldots+Q_n) \end{aligned}$$

where

$$Q_n = \left\{-\left(1-\frac{1}{1.n}\right)\frac{x^2}{2}+\frac{x^2}{2}-\left(1-\frac{1}{1.n}\right)\left(1-\frac{1}{2.n}\right)\frac{x^3}{3}+\frac{x^3}{3}\right.$$
$$\left.-\left(1-\frac{1}{1.n}\right)\left(1-\frac{1}{2.n}\right)\left(1-\frac{1}{3.n}\right)\frac{x^4}{4}+\frac{x^4}{4}+\ldots\right\}$$

Hence

$$\begin{aligned} |Q_n| &\le \frac{|x|^2}{2}\left(1-\left(1-\frac{1}{1.n}\right)\right)+\frac{|x|^3}{3}\left(1-\left(1-\frac{1}{1.n}\right)\left(1-\frac{1}{2.n}\right)\right)+\ldots \\ &= \sum_{r=2}^{\infty}\left\{\frac{|x|^r}{r}\left(1-\left(1-\frac{1}{1.n}\right)\left(1-\frac{1}{2.n}\right)\ldots\left(1-\frac{1}{(r-1)n}\right)\right)\right\} \\ &\le \sum_{r=2}^{\infty}\frac{|x|^r}{r}\left|\left(1+\frac{1}{1.n}\right)\left(1+\frac{1}{2.n}\right)\ldots\left(1+\frac{1}{(r-1)n}\right)-1\right| \end{aligned}$$

$$\left\{\left(\text{since }\left(1+\frac{1}{1.\text{n}}\right)\left(1+\frac{1}{2.\text{n}}\right)\ldots\left(1+\frac{1}{(\text{r}-1)\text{n}}\right)\right.\right.$$
$$+\left.\left(1-\frac{1}{1.n}\right)\left(1-\frac{1}{2.n}\right)\ldots\left(1-\frac{1}{(r-1)n}\right)\ge 2\right)$$

and so

$$1-\left(1-\frac{1}{1.n}\right)\ \left(1-\frac{1}{2.n}\right)\ldots\left(1-\frac{1}{(r-1)n}\right)$$

$$
\begin{aligned}
&\le 1-\left\{2-\left(1+\frac{1}{1.n}\right)\left(1+\frac{1}{2.n}\right)\dots\left(1+\frac{1}{(r-1)n}\right)\right\}\right\} \\
&\le \sum_{r=2}^{\infty}\frac{x^r}{r}\left\{\frac{1}{n}\sum_{j\le r}\frac{1}{j}+\frac{1}{n^2}\left(\sum_{j\le r}\frac{1}{j}\right)^2+\dots\text{to r terms}\right\} \\
&\le \sum_{r=2}^{\infty}\frac{x^r}{r}\left\{\frac{1}{n}\left(\log r+O(1)\right)+\frac{1}{n^2}\left(\log r+O(1)\right)^2+\dots\text{to r terms}\right\} \\
&= \sum_1+\sum_2
\end{aligned}
$$

where $\sum_1$ is over those r with $n>(10\log r+O(1))^2$ *i.e.* $(\log r+O(1))\le 5\sqrt{n}$ and $\sum_2$ those with the remaining r namely $r>N, N=exp(\frac{n}{10}+O(1))$.

$$
\begin{aligned}
\sum_1 &\le \sum_{r=2}^{\infty}\frac{|x|^r}{r}\left\{\frac{1}{n}5\sqrt{n}+\frac{1}{n^2}(5\sqrt{n})^2+\text{ to r terms}\right\} \\
&\le \sum_{r=2}^{\infty}\frac{|x|^r}{r}\left\{\frac{5}{\sqrt{n}}+(\frac{5}{\sqrt{n}})^2+(\frac{5}{\sqrt{n}})^3+\ \dots\ to\ \infty\right\} \\
&= \sum_{r=2}^{\infty}\frac{|x|^r}{r}\left\{\frac{\frac{5}{\sqrt{n}}}{1-\frac{5}{\sqrt{n}}}\right\}=\sum_{r=2}^{\infty}\frac{|x|^r}{r}\left\{\frac{5}{\sqrt{n}-5}\right\} \\
&\le \sum_{r=2}^{\infty}\frac{|x|^r}{r}.\frac{10}{\sqrt{n}}\ if\ n\ \sqrt{n}-5\ge\frac{\sqrt{n}}{2}\ i.e.\ if\ n\ge 100. \\
&\le \frac{|x|^2}{1-|x|}.\frac{10}{\sqrt{n}}.
\end{aligned}
$$

Next

$$
\begin{aligned}
\sum_2 &\le \frac{1}{N}\sum_{r\ge N}|x|^r\left(1+\frac{1}{n}\right)^{r-1} \text{ see the third step of inequality for} |\mathrm{Q_n}| \\
&\le \frac{1}{N}\sum_{r\ge 0}|x|^r\left(1+\frac{1}{n}\right)^{r-1} \\
&\le \frac{n}{N}\left(1+\frac{1}{n}\right)^{-1}\left\{1-|x|\left(1+\frac{1}{n}\right)\right\}^{-1} \\
&= \frac{2n}{N}\left\{1-|x|\left(1+\frac{1}{n}\right)\right\}^{-1} \text{ provided } |x|\left(1+\frac{1}{\mathrm{n}}\right)<1, \\
&\le N^{-\frac{1}{2}}\left\{1-|x|\left(1+\frac{1}{n}\right)\right\}^{-1} \text{ since } \mathrm{N}=exp\left(\frac{n}{10}+O(1)\right).
\end{aligned}
$$

Thus we have proved the following

THEOREM. *Let* $|x|<1$ *and*

$$n\left((1-x)^{\frac{1}{n}}-1\right)+\sum_{k=1}^{\infty}\frac{x^k}{k}=E(x).$$

Then

$$|E(x)| \leq \frac{10|x|^2}{1-|x|}\ n^{-\frac{1}{2}} + N^{-\frac{1}{2}}\left\{1-|x|\left(1+\frac{1}{n}\right)\right\}^{-1}$$

where $n \geq 100$ *and* $N = exp(\frac{n}{10} + O(1))$.

COROLLARY. Letting $n \to \infty$ we have for $|x| < 1$, the equality

$$\lim_{n\to\infty} n\left((1-x)^{\frac{1}{n}} - 1\right) = -\sum_{k=1}^{\infty} \frac{x^k}{k} = \log(1-x)\,.$$

REMARK 1. An alternative approach is to assume an expansion of the following type. (In this approach binomial theorem for a non-integral positive index is not necessary. But we need "multinomial theorem" for a positive integral index). We start with

$$\left(1+\frac{z}{n}\right)^n = e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \frac{1}{1-x}\ (|x| < 1)$$

$$e^z = \frac{1}{1-x} = 1 + x + x^2 + \ldots |x| < 1$$

Assume that $z = a_0 + a_1 x + a_2 x^2 + \ldots$. There follows

$$\frac{1}{1-x} = 1 + \frac{(a_0 + a_1 x + a_2 x^2 + \ldots)}{1!} + \frac{(\ldots)^2}{2!} + \frac{(\ldots)^3}{3} + \ldots.$$

Equating constant terms we get $1 = e^{a_0}$ and so $a_0 = 0$. Equating coefficients of x we have $1 = a_1$. Equating coefficients of x^2 we have $1 = \frac{a_2}{1!} + \frac{a_1^2}{2!} i.e.\ a_2 = \frac{1}{2}$ and so on. By induction we may complete the proof that

$$\log\left(\frac{1}{1-x}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

But we will not pursue this proof and do error estimates.

REMARK 2. Yet another method is to use

$$\begin{aligned}(1-x)^y &= 1 - xy + (-1)^2 x^2\ \frac{y(y-1)}{2!} + (-1)^3 x^3\ \frac{y(y-1)(y-2)}{3!} + \ldots \\ &= Exp(y(\log(1-x))) = 1 + y\ \log(1-x) + \ldots\end{aligned}$$

and equate coefficients of y on both sides and thus

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$$

This method is explained in a chapter of the book: Algebra by HALL & KNIGHT, VOLUME 1.

REMARK 3. The theorem gives in addition the error estimate.

CHAPTER - VII

(TRIGONOMETRIC FUNCTIONS AND THEIR INVERSES)

We start with the diagram of triangle ABC

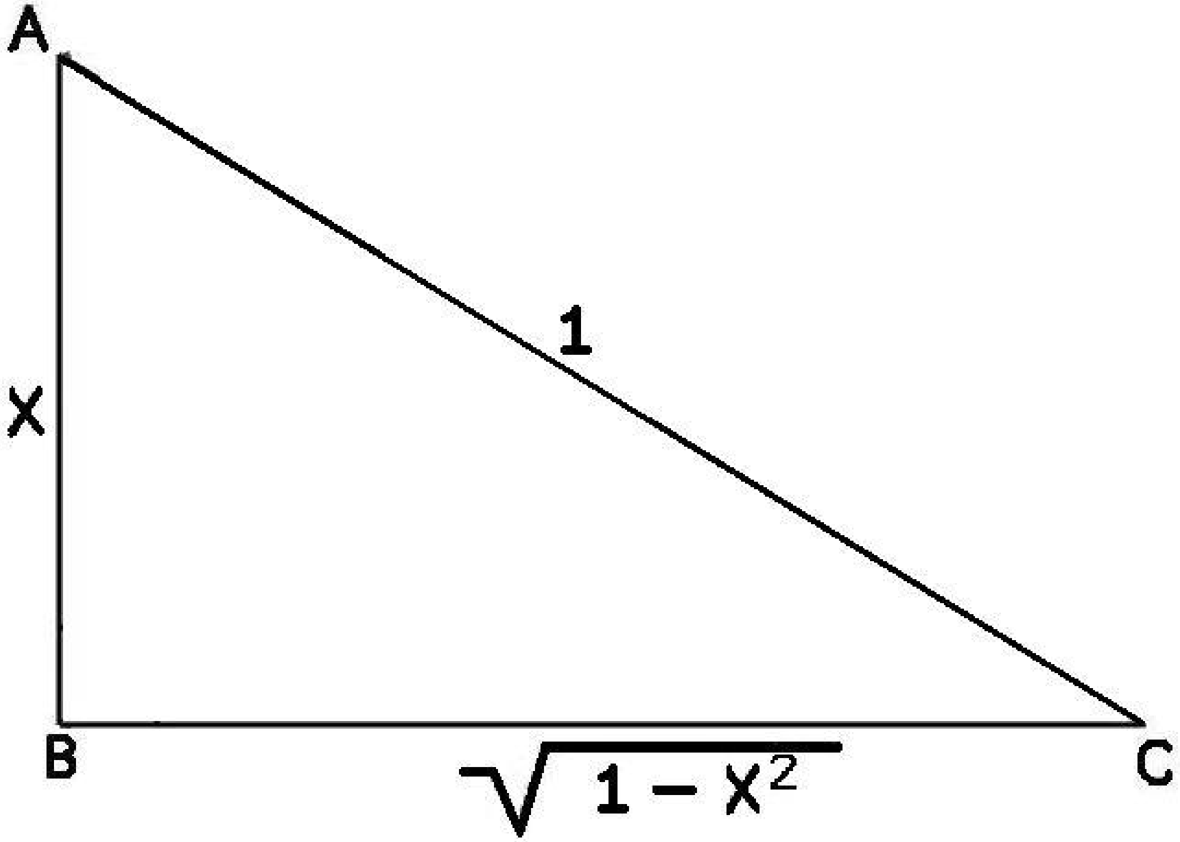

Figure 1

where angle B (denoted by $\hat{B}$) $= \frac{\pi}{2}$. It is clear that $sin^{-1}x = \hat{C}, cos^{-1}\ x + sin^{-1}x = \hat{A} + \hat{C} = \frac{\pi}{2}, tan^{-1}\frac{x}{\sqrt{1-x^2}} = \hat{C},\ cot^{-1}\frac{x}{\sqrt{1-x^2}} = \hat{A}$ and so $tan^{-1}\frac{x}{\sqrt{1-x^2}} + cot^{-1}\frac{x}{\sqrt{1-x^2}} = \hat{C} + \hat{A} = \frac{\pi}{2}$ and so on. These require the condition $0 < x < 1$ (but relaxable by "Analytic continuation", a term which we do not explain).

From

$$cos\ \theta + i\ sin\ \theta = e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$$

Follows (on equating real and imaginary points),

$$cos\ \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - + \dots \text{and}$$

$$sin\ \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \dots$$

The usual (nice) series for $tan^{-1}x$ can be obtained as follows.

We have

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Put $z = x + iy$ and we get

$$-\log(1-x-iy) = \sum_{n=1}^{\infty} \frac{(x+iy)^n}{n}.$$

Specializing this to $x = 0$ we obtain

$$-\log(1-iy) = \sum_{n=1}^{\infty} \frac{(iy)^n}{n}.$$

Here LHS $= -\log|1-iy| - \text{i } \tan^{-1}(-y) = -\log|1-iy| + \text{i } \tan^{-1}\text{y}$ Thus

$$-\log(1-iy) = \sum_{n=1}^{\infty} \frac{(iy)^n}{n} = -\log|1-iy| + i\ tan^{-1}y$$

Equating imaginary points in the last two formulae we get

$$tan^{-1}y = y - \frac{y^3}{3} + \frac{y^5}{5} - + \dots, \text{where} |y| < 1.$$

But $tan^{-1}1 = \frac{\pi}{4}$ and we recover

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \dots.$$

by justifying the limit operation $y \to 1$.

So far we have not employed calculus. But we now use calculus.

$$\frac{d}{dy} tan^{-1}y = \frac{1}{1+y^2} \text{ and so } tan^{-1}y = \int_0^y \frac{d}{dy}\,(tan^{-1}y)dy \text{ and so for } |y| < 1$$

$$= \int_0^y (1 - y^2 + y^4 - \dots)dy = y - \frac{y^3}{3} + \frac{y^5}{5} - + \dots$$

Again $\frac{d}{dy}\ sin^{-1}y = \frac{1}{\sqrt{1-y^2}}$ and so $sin^{-1}y = \int_0^y \frac{dy}{\sqrt{1-y^2}}$

$$= \int_0^y \left(1 + \left(-\frac{1}{2}\right)\frac{(-y^2)}{1!} + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\frac{(-y^2)^2}{2!} + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\frac{(-y^2)^3}{3} + \dots\right) dy$$

$$= y + \frac{1}{2}\frac{y^3}{3} + \frac{1.3}{2.4}\ \frac{y^2}{5} + \frac{1.3.5}{2.4.6}\ \frac{y^7}{7} + \dots.$$

Also $cos^{-1}y = \frac{\pi}{2} - sin^{-1}y = \frac{\pi}{2} - (y + \frac{1}{2}\frac{y^3}{3} + \frac{1.3}{2.4}\ \frac{y^5}{5} + \ldots)$.

We end this book by proposing a new method for a nice expansion of $(F(x))^k$ where k is any positive integer constant. We limit ourselves to $(tan^{-1}x)^2$ and $(sin^{-1}x)^2$. [Before proceeding further we note (p.203 of part II of the excellent books by S.L.Loney (parts I and II)) the following result.

$$\frac{1}{6}(sin^{-1}x)^3 = \frac{1}{2}\frac{x^3}{3} + \left(\frac{1}{1^3} + \frac{1}{3^3}\right)\ \frac{1.3}{2.4}\ \frac{x^5}{5} + \left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3}\right)\ \frac{1.3.5}{2.4.6}\ \frac{x^7}{7} + \ldots$$

We hope that we can obtain this by our method].

We now proceed with our method.

Let

$$(tan^{-1}x)^2 = a_0 + a_1x + a_2x^2 + \ldots.$$

Differentiating both sides with respect to x and multiplying both sides by $1 + x^2$ we get

$$\begin{aligned} 2\ tan^{-1}x &= \left(1 + x^2\right)\left(a_1 + 2a_2x + 3a_3x^2 + \ldots\right) \\ &= 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - + \ldots\right) \end{aligned}$$

Equating coefficients of like powers of x, we can obtain $a_0, a_1, a_2, \ldots$. Certainly $a_0 = 0$. But in this special case we can get $a_0, a_1, a_2, \ldots$ by direct squaring in

$$\left(x - \frac{x^3}{3} + \frac{x^5}{5} - + \ldots\right)^2.$$

We now turn to

$$(sin^{-1}x)^2 = a_0 + a_1x + a_2x^2 + \ldots \tag{1}$$

differentiating we get

$$\frac{2sin^{-1}x}{\sqrt{1 - x^2}} = a_1 + 2a_2x + 3a_3x^2 + \ldots \tag{2}$$

One more differentiation gives

$$\frac{2}{1 - x^2} + 2(sin^{-1}x)(-\frac{1}{2}(1 - x^2)^{-\frac{3}{2}}(-2x)) = 2.1.a_2 + 3.2.a_3x + 4.3.a_4x^2 + \ldots \tag{3}$$

Multiplying throughout by $1 - x^2$, we obtain

$$2 + \frac{2(sin^{-1}x)}{\sqrt{1 - x^2}} = (1 - x^2)(2.1.a_2 + 3.2.a_3x + 4.3.a_4x^2 + \ldots$$

Here we substitute for LHS (using (2))

$$2 + a_1 + 2a_2 x + 3a_3 x^2 + \dots$$
$$= \left(1 - x^2\right)\left(2.1.a_2 + 3.2.a_3 x + 4.3.a_4 x^2 + \dots\right). \qquad (4)$$

In (4) we equate coefficients of like powers of x and we get $a_1, a_2, a_3, \dots$ (trivially $a_0 = 0$ and $a_1 = 0$).

We leave it as an exercise for the reader to complete the expansion of $(sin^{-1}x)^2$.

ONE FINAL REMARK: JONATHAN M BORWEIN and MARC CHAMBERLAND have proved the following surprising result.

THEOREM: (1) for $|x| \leq 2$ and $N = 1, 2, 3, \dots$ we have

$$\frac{1}{(2N)!}\left(sin^{-1}\frac{x}{2}\right)^{2N} = \sum_{k=1}^{\infty} \frac{H_N(k)}{\binom{2k}{k} k^2} x^{2k}$$

where $H_1(k) = \frac{1}{4}$ and

$$H_{N+1}(k) = \frac{1}{4}\sum_{n_1=1}^{k-1} \frac{1}{(2n_1)^2} \sum_{n_2=1}^{n_1-1} \frac{1}{(2n_2)^2} \cdots \sum_{n_N=1}^{n_{N-1}-1} \frac{1}{(2n_N)^2}.$$

(2) For $|x| \leq 2$ and $N = 0, 1, 2, \dots$ we have

$$\frac{1}{(2N+1)!}\left(sin^{-1}\ \frac{x}{2}\right)^{2N+1} = \sum_{k=0}^{\infty} \frac{G_N(k)\binom{2k}{k}}{2(2k+1)4^{2k}}\ x^{2k+1}$$

where $G_0(k) = 1$ and

$$G_N(k) = \sum_{n_1=0}^{k-1} \frac{1}{(2n_1+1)^2} \sum_{n_2=0}^{n_1-1} \frac{1}{(2n_2+1)^2} \cdots \sum_{n_N=0}^{n_{N-1}-1} \frac{1}{(2n_N+1)^2}.$$

The convention is that the sum is zero if the starting index exceeds the finishing index.

The beautiful result

$$\sum_{k=1}^{\infty} \frac{4^k}{\binom{2k}{k} k^3} = \pi^2\ \log 2 - \frac{7}{8}\,\zeta(3)$$

is deduced in their paper as a special case.

REFERENCE

[1] S.L.LONEY, Plane Trigonometry volume I, Essential books 4393/4, 1st floor, Tulsidas Street, Ansari Road, Daryaganj, New Delhi - 110 002.

[2] S.L.LONEY, Plane Trigonometry Volume II, Publishers same as in [1].

[3] K.RAMACHANDRA, Pythagoras' theorem and similar triangles, Math. Gazette, Vol. 86, No. 506, July 2002, p.324.

A reference book which we have not seen is

[4] I M GELFAND and M.SAUL, Trigonometry, Birkhauser (Boston), 1st edition (June 8, 2001).

The reference to the surprising result mentioned in the end is

[5] JONATHAN M.BORWEIN and MARC CHAMBERLAND, Integer Powers of arc sin, International Journal of Mathematics and Mathematical Sciences, volume 2007, Article ID 19381, 10 pages, doi: 10.1155/2007/19381.

We have every reason to believe that [4] is a very good book.

National Institute of Advanced Studies,
Indian Institute of Science Campus,
Bangalore-560 012.

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EPILOGUE

The reader who is interested only in series for sines and cosines may stop after reading the first five chapters of the book.